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Damping of the free oscillations of a neutral buoyancy sphere in a viscous stratified fluid☆

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ABSTRACT

A technique is developed for calculating the oscillations of balanced spheres at neutral buoyancy levels based on the linearization of the equations of the mechanics of a viscous, continuously stratified fluid. A self-consistent system of integro-differential equations is obtained and analysed using perturbation theory methods. The results of calculations of the displacements of the centres of the spheres are reduced to a form which a permits direct comparison with a laboratory experiment and they agree with the data of measurements. A comparison is made with calculations of the free oscillations of a sphere in an ideal fluid.

When analysing the results of measurements obtained using profiling and freely drifting neutral buoyancy buoys, it is assumed that they do not disturb the structure of the stratified medium. The amplitude-frequency characteristics when installing a buoy at neutral buoyancy levels is determined in the approximation of an ideal, continuously stratified[1,2] fluid and a two-layer[3] fluid. However, the results of calculations of the oscillations of probes about the neutral buoyancy level are noticeably different from the data from laboratory[4] and semi-natural measurements.[5] Schlieren shadow flow visualization patterns and detailed trajectory measurements of the oscillations of bodies which are freely sinking to the neutral buoyancy level in a continuously stratified medium shows that both internal waves as well as the non-wave components of flows, including boundary layers and short-lived extended vortices, have an effect on their motion.[6,7] In this connection, a detailed analysis of the processes by which a sphere becomes established at a neutral buoyancy level taking into account the radiation of internal waves and the dissipation effect is of interest.

1. The system of governing equations

A stable, exponentially stratified fluid is considered, the density of which $\rho_0(z)=\rho_0\exp(-z'/\Lambda)$, in the system of coordinates (z', y', z') associated with the fluid, is characterized by a scale Λ, frequency n and a buoyancy period T_b (the z' axis is directed opposite to the acceleration of the force of gravity $\mathbf{g}$, ρ_{00} is the density at the neutral buoyancy level and $z'=0$, $N=\sqrt{g/\Lambda}=2\pi/T_b$.

The equation of motion of a sphere of mass m, on which a buoyancy force F_A and a wave force F_z act, has the form[3,8]

$$m\zeta'' = F_A + F_z \tag{1.1}$$

In the case of small displacements ζ (with respect to the sphere radius R, the initial displacement ζ_0 and the buoyancy scale Λ), the buoyancy force is equal to

$$F_A = -g\int\limits_0^{\pi}\rho_{00}\pi R^3\sin^3\theta\left(\exp\left(-\frac{z'}{\Lambda}\right)-1\right)d\theta \approx -\frac{4\pi R^3}{3}\rho_{00}N^2\zeta$$

where θ is the polar angle in a spherical system of coordinates associated with the centre of the sphere.

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In a Cartesian system of coordinates (x, y, z) associated with the centre of the sphere, the dynamic component of the force, which is determined by the distribution of the pressure P and the viscous stress tensor, has the form[9]

$$F_z = -\rho\oint\left\{-\frac{P}{\rho}n_z + \nu\left[2\frac{\partial \upsilon_r}{\partial r}n_z + \left(\frac{\partial \upsilon_z}{\partial r} + \frac{\partial \upsilon_r}{\partial z}\right)n_r\right]\right\}dS; \quad n_z = \cos\theta, \quad n_r = \sin\theta$$

where υ_r, υ_φ and υ_z are the radial, azimuthal and vertical components of the fluid velocity in a cylindrical system of coordinates and v is the coefficient of kinematic viscosity. By virtue of the symmetry of the problem, the radial component of the force is equal to zero.

At the initial instant $t=0$, both the fluid and the sphere, which is separated by a distance ζ_0 from the neutral buoyancy level $z=0$, are assumed to be motionless:

$$\mathbf{v} = 0, \quad \zeta = \zeta_0, \quad \frac{d\zeta}{dt} = 0 \quad \text{when} \quad t = 0 \tag{1.2}$$

Equation (1.1) with boundary conditions (1.2) is solved simultaneously with the equations of motion of a viscous incompressible stratified fluid written in the laboratory system of coordinates[9] a (x', y', z')

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla\tilde{P} + \rho\nu\Delta\mathbf{v} - \tilde{\rho}g\mathbf{e}_z,$$
$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{v}) = 0, \quad \operatorname{div}\mathbf{v} = 0, \quad \rho = \rho_0(z') + \tilde{\rho} \tag{1.3}$$

Here, $\mathbf{v}=(\upsilon_x, \upsilon_y, \upsilon_z)$ is the fluid velocity, $\mathbf{e}_z$ is the unit vector, the effects of diffusion are neglected, $\tilde{P}$ is the pressure after subtracting the hydrostatic pressure, and $\tilde{\rho}$ is the density perturabation caused by the oscillations of the sphere.

The boundary conditions for system (1.3) include the condition for the decay of the perturbations at infinity and no-slip condition at the surface $\Gamma_s = x'^2 + y'^2 + (z' - \zeta)^2 - R^2 = 0$ of a sphere of radius R, moving according to the law $z'_s(t) = \zeta(t)$:

$$\mathbf{v}|_{\Gamma_s} = \left(0, 0, \frac{\partial\zeta}{\partial t}\right) \tag{1.4}$$

At long times, when the sphere stops at the neutral buoyancy level, the fluid motions also attenuate in the whole space:

$$\left(\mathbf{v}, \tilde{P}, \tilde{\rho}\right) = 0, \zeta = 0, \frac{d\zeta}{dt} = 0 \quad \text{when} \quad t \to \infty \tag{1.5}$$

In the system of coordinates with its origin at the centre of mass of the sphere (x, y, z), which is related to the laboratory system by the relations

$$x' = x, \quad y' = y, \quad z' = z + \zeta(t)$$

system of equations (1.3) and boundary conditions (1.4) and (1.5) take the form

$$\rho\left(\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla\tilde{P} + \rho\nu\Delta\mathbf{v} - \tilde{\rho}g\mathbf{e}_z - \rho\frac{d^2\zeta}{dt^2}\mathbf{e}_z, \quad \operatorname{div}\mathbf{v} = 0$$
$$\rho = \rho_0(z,t) + \tilde{\rho}, \quad \rho_0(z,t) = \rho_{00}\exp\left(-\frac{z + \zeta(t)}{\Lambda}\right), \quad \frac{\partial\tilde{\rho}}{\partial t} = \frac{\rho_0}{\Lambda}\left(\upsilon_z + \frac{d\zeta}{dt}\right)$$
$$\mathbf{v}|_{\Gamma_s} = 0 \tag{1.6}$$

In deriving Eqs (1.6), unlike the well-known approach,[1] account has been taken of the fact that, in the system of coordinates (x, y, z) associated with the sphere, the distribution of the density ρ_0 of the stratified fluid is a function of time. System (1.6) belongs to the class of singularly perturbed systems (small coefficients for the leading derivatives), and the solution of such systems is found using a well-known technique.[10]

2. Construction of the solution of the governing system

In the linear approximation, the system of equations (1.6) becomes the well known internal wave equation[11]

$$-\frac{1}{\Lambda}\frac{d\zeta}{dt}\left(\hat{L}\upsilon_z - \frac{1}{\Lambda}\frac{\partial^2\upsilon_z}{\partial t\partial z}\right) + \hat{L}\upsilon_z - \frac{1}{\Lambda}\frac{\partial^3\upsilon_z}{\partial t^2\partial z} + N^2\Delta_\perp\upsilon_z = 0$$
$$\hat{L} = \left(\frac{\partial}{\partial t} - \nu\Delta\right)\Delta, \quad \Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \Delta = \Delta_\perp + \frac{\partial^2}{\partial z^2} \tag{2.1}$$

In the Boussinesq approximation, neglecting the viscosity effects and terms of the order of $d\zeta/dt$, Eq. (2.1) becomes a well-known reduced equation.[1]

By taking account of the symmetry of the sphere and the internal wave field we can simplify Eq. (2.1) by introducing a stream function Ψ, which defines the radial $\upsilon_r = r^{-1}\partial\Psi/\partial z$ and vertical $\upsilon_z = r^{-1}\partial\Psi/\partial r$ components of the velocity[9] in the cylindrical system of coordinates (r, φ, z). By virtue of the symmetry of the problem, the azimuthal component of the fluid velocity is identically equal to zero.

Axi, symmetric solutions of Eq. (2.1), containing the time-dependent factor $d\zeta/dt$, are found by expansion in Fourier - Bessel integrals with respect to the variables k and κ, that is, with respect to the vertical and radial components of the wave number (everywhere below, unless otherwise stated, integration with respect to k is carried out from $-\infty$ to ∞ and, with respect to κ, from 0 to ∞)

$$\upsilon_z(r,z,t)=\iint f(k,\kappa,t)\,e^{ikz}J_0(\kappa r)\,d\kappa\,dk \tag{2.2}$$

where $J_n(\kappa r)$ is a Bessel function of the first kind of order n.

Substituting the expansion (2.2) into Eq. (2.1) we obtain the equation for the spectral velocity function $f=f(k,\kappa,t)$

$$\mu\frac{\partial^2 f}{\partial t^2}+\left(\nu k_0^2-\frac{\mu}{\Lambda}\frac{d\zeta}{dt}\right)\frac{\partial f}{\partial t}+\frac{\kappa^2N^2}{k_0^2}f-\frac{\nu k_0^2}{\Lambda}\frac{d\zeta}{dt}f=0$$
$$\mu=1+\frac{ik}{\Lambda k_0^2},\quad k_0^2=k^2+\kappa^2 \tag{2.3}$$

which determines the radial component of the velocity and the stream function Ψ.

Taking account of the incompressibility condition and the rules for the differentiation and integration of Bessel functions[12] we can obtain expressions for the radial component of the velocity and the stream function

$$\upsilon_r(r,z,t)=-i\iint f(k,\kappa,t)e^{ikz}\frac{kJ_1(\kappa r)}{\kappa}d\kappa dk,\quad \Psi(r,z,t)=\iint f(k,\kappa,t)e^{ikz}\frac{J_1(\kappa r)}{\kappa}d\kappa dk$$

The value of the pressure is found by integrating Eq. (1.3) using representation (2.2):

$$P=-i\rho\iint\left(\frac{\partial f}{\partial t}+\nu k_0^2 f\right)e^{ikz}\frac{J_0(\kappa r)}{k}d\kappa\,dk$$

For brevity, the calculated vertical component of the total force acting on the sphere is not presented here.

In the dimensionless variables $\xi=kR$ and $\eta=\kappa R$, when account is taken of the calculated values of the forces, the expansions of the trigonometric functions occurring in them and the integrals of them in series in Bessel functions, Eq. (1.1) takes the form

$$\zeta''+N^2\zeta+\frac{1}{R}\left\{\int_0^\infty\int_0^\infty G_1(\xi,\eta)\frac{\partial f}{\partial t}d\xi d\eta+\nu\int_0^\infty\int_0^\infty G_2(\xi,\eta)f\,d\xi d\eta\right\}=0$$
$$G_1(\xi,\eta)=-4\sum_{n=1}^{\infty}\frac{J_{2n-1}(\xi)}{\xi}\sum_{m=0}^{n}(-1)^m C_{2n-1}^{2m}m!\,2^m(2(n-m)-1)!!\sum_{p=0}^{m}\frac{\eta^{2p}C_m^p}{2^p p!}\left(\frac{1}{\eta}\frac{d}{d\eta}\right)^{p+n}\frac{\sin\eta}{\eta}$$
$$G_2(\xi,\eta)=\frac{2}{\eta}\left(\frac{\sin\eta}{\eta}-\cos\eta\right)J_0(\xi)+$$
$$+4\sum_{n=1}^{\infty}J_{2n}(\xi)\sum_{m=0}^{n}(-1)^m C_{2n}^{2m}(m+1)!\,2^m(2(n-m)-1)!!\sum_{p=0}^{m}\frac{\eta^{2p}C_m^p}{2^p(p+1)!}\left(\frac{1}{\eta}\frac{d}{d\eta}\right)^{p-n+1}\frac{\sin\eta}{\eta} \tag{2.4}$$

In low viscosity, weakly stratified fluids, the governing dimensionless complexes, given by the relations

$$\frac{z}{\Lambda}\sim\frac{\zeta_0}{\Lambda}\sim\frac{R}{\Lambda}\sim 10^{-3},\quad \varepsilon=\frac{1}{\mathrm{St}}=\frac{\nu}{NR^2}\sim 10^{-3},\quad \alpha=\frac{\zeta_0}{\varepsilon\Lambda}\sim 0.1,\quad \beta=\frac{R}{\varepsilon\Lambda}\sim 0.1$$

(St is the Stokes number), are found to be small quantities and will be used as expansion parameters in the subsequent analysis. Taking account of the existence of small factors of a different nature, the method of multiple scales is used as the basic instrument of the analysis.

When account is taken of the notation introduced for the small parameters, Eqs (2.3) and (2.4) are transformed into the system of integro-differential equations

$$\frac{\partial^2 W}{\partial\tau^2}+\left[\frac{\alpha\varepsilon}{2}\frac{d^2\Xi}{d\tau^2}+\frac{\lambda^2-\alpha\varepsilon^2(\xi^2+\eta^2)\frac{d\Xi}{d\tau}}{\chi}-\frac{\varepsilon^2}{4\chi^2}\left(\xi^2+\eta^2+\alpha\chi\frac{d\Xi}{d\tau}\right)^2\right]W=0$$
$$\frac{d^2\Xi}{d\tau^2}+\Xi+\exp\left(\frac{\alpha\varepsilon}{2}\Xi\right)\left[\iint G(\xi,\eta,\tau)\frac{\partial W}{\partial\tau}d\xi d\eta\right.$$
$$\left.+\frac{\varepsilon}{2}\iint Q(\xi,\eta,\tau)W\,d\xi d\eta+\frac{\alpha\varepsilon}{2}\frac{d\Xi}{d\tau}\iint G(\xi,\eta,\tau)W\,d\xi d\eta\right]=0 \tag{2.5}$$

which relate the dimensionless spectral velocity function

$$W=\frac{f}{N\zeta_0R^2}\exp\left(\gamma t-\frac{\alpha\varepsilon\Xi}{2}\right)$$

to the relative displacement $\Xi = \zeta/\zeta_0$, where

$$\tau = Nt, \quad \lambda^2 = \frac{\eta^2}{\xi^2+\eta^2}, \quad \gamma = \varepsilon\frac{\xi^2+\eta^2}{2\chi}, \quad \chi = 1 + \frac{i\varepsilon\beta\xi}{\xi^2+\eta^2}. \quad G(\xi,\eta,\tau) = e^{-\gamma t}G_1(\xi,\eta)$$

$$Q(\xi,\eta,\tau) = e^{-\gamma t}[2G_2(\xi,\eta) - \gamma G_1(\xi,\eta)]$$

The function f is the solution of Eq. (2.3).

System (2.5) is considered with the initial conditions

$$W = 0, \quad \Xi = 1, \quad \frac{d\Xi}{d\tau} = 0 \quad \text{when} \quad \tau = 0$$

and the condition for the attensation of all motions at long times

$$W = 0, \quad \Xi = 0, \quad \frac{d\Xi}{d\tau} = 0 \quad \text{when} \quad \tau \to \infty$$

The basic properties of solution (2.2) are determined by the method of multiple scales,[13] taking account of the smallness of the ratios σ, α and β.

The set of time parameters

$$T_0 = T, \quad T_1 = \sigma T, \quad T_2 = \alpha T, \quad T_3 = \beta T, \quad T_4 = \alpha\sigma T,$$
$$T_5 = \beta\sigma T, \quad T_6 = \alpha\beta T, \quad T_7 = \sigma^2 T, \quad \dots$$

is introduced in the implementation of this method.

Time derivatives are then represented in the form of the expansions

$$\frac{d}{dT} = D_0 + \sigma D_1 + \alpha D_2 + \beta D_3 + \alpha\sigma D_4 + \dots$$

$$\frac{d^2}{dT^2} = D_{00} + 2\sigma D_{01} + 2\alpha D_{02} + 2\beta D_{03} + 2\alpha\sigma(D_{04} + D_{12}) + \dots$$

where

$$D_0 = \frac{\partial}{\partial T_0}, \ D_k = \frac{\partial}{\partial T_k}, \ D_{00} = \frac{\partial^2}{\partial T_0^2}, \ D_{0k} = \frac{\partial^2}{\partial T_k^2}; \quad k = 1, 2, \dots$$

The functions W and Ξ are represented in a similar manner:

$$W = W_0 + \sigma W_1 + \alpha W_2 + \beta W_3 + \alpha\sigma W_4 + \dots$$
$$\Xi = \Xi_0 + \sigma\Xi_1 + \alpha\Xi_2 + \beta\Xi_3 + \alpha\sigma\Xi_4 + \dots \tag{2.6}$$

The functions $G = G(\xi, \eta, T_1)$ and $Q = Q(\xi, \eta, T_1)$ depend solely dependent on the parameter T_1 since the factor

$$\exp\left[\frac{\sigma}{\chi}(\xi^2+\eta^2)t\right] = \exp\left[\frac{T_1}{\chi}(\xi^2+\eta^2)\right]$$

appears in the representation of the kernels G and Q.

Substituting expansions (2.6) into system (2.5), we obtain new systems of equations for determining of the functions W_i and Ξ_i, for the clear writing of which we introduce the following notation:

$$H_0 = \iint G(T_1)D_0W_0 dS, \quad H_k = \iint G(T_1)(D_0W_k + D_kW_0)dS$$

The zeroth-order system has the form

$$D_{00}W_0 + \lambda^2 W_0 = 0, \quad D_{00}\Xi_0 + \Xi_0 + H_0 = 0 \tag{2.7}$$

The corresponding systems of the first order with respect to the parameters σ, α and β are

$$D_{00}W_1 + \lambda^2 W_1 = -2D_{10}W_0, \quad D_{00}\Xi_1 + \Xi_1 + H_1 = -2D_{10}\Xi_0$$
$$D_{00}W_0 + \lambda^2 W_2 = -2D_{02}W_0, \quad D_{00}\Xi_2 + \Xi_1 + H_2 = -2D_{02}\Xi_0 - \iint Q(T_1)W_0 dS$$
$$D_{00}W_3 + \lambda^2 W_3 = -2D_{03}W_0, \quad D_{00}\Xi_2 + \Xi_1 + H_2 = -2D_{03}\Xi_3 \tag{2.8}$$

Taking account of the initial conditions, we will seek the solution of system (2.7) in the form

$$W_0 = A(\xi,\eta)\sin(\lambda T_0) \tag{2.9}$$

Substituting expression (2.9) into the second equation of system (2.7), we obtain an equation, the solution of which is found using standard methods and has the form

$$\Xi_0 = -\iint G(T_1)A(\xi,\eta)\cos(\lambda T_0)\frac{\lambda}{1-\lambda^2}dS \tag{2.10}$$

The change of variables

$$\xi = r_\rho\cos\varphi,\quad \eta = r_\rho\sin\varphi,\quad r_\rho\in[0,\infty),\quad \varphi\in[0,\pi/2]$$

is introduced to simplify the expression for Ξ_a. In this case, $dS = r_\rho dr_\rho d_\varphi$, $\lambda = \sin\varphi$. Using the representation[12]

$$\cos(T_0\sin\varphi) = J_0(T_0) + 2\sum_{k=1}^{\infty}(-1)^k J_{2k}(T_0)\cos(2k\varphi)$$

we obtain

$$\Xi_0 = \sum_{k=0}^{\infty}\gamma_k(T_1)J_{2k}(T_0)$$

$$\gamma_k(T_1) = -\int_0^\infty\int_0^{\pi/2} G(r_\rho,\varphi,T_1)A(r_\rho,\varphi)\cos(2k\varphi)\frac{\lambda}{1-\lambda^2}r_\rho dr_\rho d\varphi\times\begin{cases}1, & k=0\\ 2, & k\neq 0\end{cases} \tag{2.11}$$

The Bessel functions $J_{2k}(T_0)$ describe the oscillations of the sphere, and the functions γ_k play the role of an envelope, and they characterize the attenuation of each harmonic $J_{2k}(T_0)$ due to viscous friction and losses by the radiation of internal waves. The attenuation rate can be determined by numerical integration and depends slightly on the value of k.

For the purposes of the subsequent analysis, the expression for the displacements of the sphere is represented in the form of an expansion in a series in Bessel functions

$$\Xi_0 = \operatorname{erf}\left(\frac{aR}{\delta_N\sqrt{\tau}}\right)\sum_{m=0}^{\infty}b_m J_{2m}(\tau),\quad \delta_N = \sqrt{\frac{\nu}{N}} \tag{2.12}$$

where δ_N is a universal microscale, characterizing the thickness of the Stokes-type periodic boundary flow.[7] The values of the parameter a can be found from measurements of the damping of the free oscillations of the sphere. Preliminary processing of the experimental data showed that the basic characteristics of the pattern of oscillations are already described by the first term of series (2.12) which, for convenience, we present in the physical variables

$$\Xi_0 = \frac{\zeta}{\zeta_0} = \operatorname{erf}\left(\frac{aR}{\delta_N\sqrt{\tau}}\right)J_0(\tau)+\ldots \tag{2.13}$$

The first factor characterizes the viscosity effect, and the second factor the buoyancy, effects which are responsible for the difference in the structure of expression (2.12) from the results of calculations[1,2] based on an inviscid fluid model.

3. Experimental apparatus

Experiments were carried out in a laboratory setting including a working tank with illuminators, a mechanism for producing stratification, an optical visualization system (an IAB-458 schlieren instrument for observing the flow pattern in a vertical plane), a cathetometer for determining the initial height and the final position of the body being investigated and an auxiliary mechanism for setting the free neutral buoyancy body with a zero velocity. The tank was filled with the stratified liquid by the method of successive displacement from below using a traditional hydraulic system. The buoyancy period was determined using the schlieren instrument or an electrical conductivity probe from measurements of the natural oscillations of the liquid excited by the sinking trace behind a gas bubble which is floating up or the trace which is floating up behind a sinking crystal of salt or sugar. This direct method, which is based on the determination of the buoyancy period, enables one to measure its value with a relative error no greater than 2%.[14] A detailed description of the experimental apparatus has been presented earlier (for example, see Ref. 7).

4. Comparison of the results of calculations and measurements of the trajectory of motion of a neutral buoyancy sphere

Both the results of individual experiments as well as results averaged over several experiments were used in the data processing and for determining the coefficients in formula (2.13). All the experimental conditions were kept constant in successive experiments in order to control the reproducibility, or just one of the governing parameters was varied such as, for example, the height of free fall of the sphere to the neutral buoyancy level. Under such conditions, the results of single-type experiments could be used to increase the accuracy of the statistical data processing, since the function $\Xi_a = \zeta/\zeta_0$, that is, the ratio of the displacement of the sphere from the neutral buoyancy level to the height of free fall, is also determined in the calculations. Spline-interpolation of the data was carried out for curves averaged over a series of experiments of the same type.

A comparison of the data obtained after spline-interpolation with formula (2.13) was performed using the method of least squares. The values of the coefficient a in formula (2.13) and the oscillation frequencies of the sphere ω_s were selected from the condition for minimizing

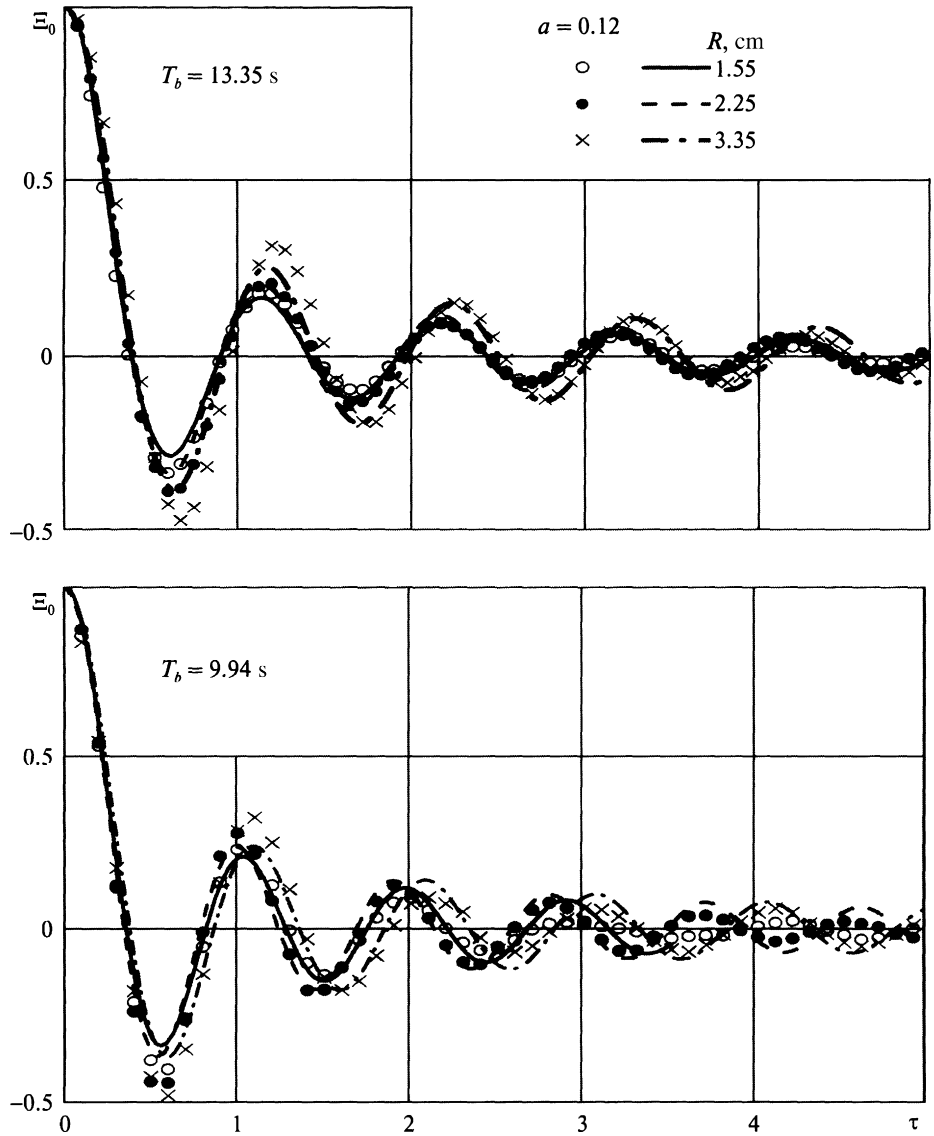

Fig. 1.

the χ^2 criterion.[15] The ratio $\Delta_N = (\omega_x - N)/N$, which characterizes the extent of the effect of induced flows and waves in the fluid on the parameters of fluid motion, was used to estimate the difference in the frequencies ω_s and the buoyancy N.

The results of a calculation using formula (2.13) (the solid, dashed and dot-dashed curves for $a = 0.12$ and the experimental data (the points) for the normalized displacements of spheres of radius R in fluids with different density gradients are shown in Fig. 1 as a function of the dimensionless time $\tau = t/T_b$ (T_b is the buoyancy period). In all cases, the sphere executes damped oscillations with a frequency close to the buoyancy frequency. The envelope of the displacements rapidly decreases during the first two to three periods of the oscillations and then varies only slightly.

The calculated and measured data are in especially good agreement everywhere in the case of spheres of minimum size ($R = 1.55$ cm, buoyancy period $T_b = 13.35$ s) apart from the first immersion of the sphere (Fig. 1). The deeper first immersion resulted from the effect of the wake with bottom and secondary vortices, which entrain fluid from the higher-lying levels, expressed at the start of the process.[16] The effects of fluid transport are not taken into account in this technique.

Quantitative comparison of the calculated and measured trajectories shows that the agreement in the first phase of rapid attenuation of the oscillations is improved if the sphere radius in the calculations is somewhat increased (by 10 - 15%). This effect is due to the influence of the boundary layer and the secondary vortices which are also neglected. In the stabilization phase of the oscillations ($\tau > 3$), all the data from the calculations shown in Fig. 1 are in satisfactory agreement with the observations.

As the buoyancy frequency decreases, the difference in the character of the oscillations of spheres of dissimilar radius (Fig. 1) shows up more clearly. The period of the oscillations increases as the radius of the sphere becomes larger, which leads to a relative displacement of the curves. As the sphere radius and the buoyancy frequency increases, the contribution from non-wave effects (secondary vortices or boundary layers) becomes greater which shows up in the discrepancy between the experimental data and the calculations.

The value of the fitting parameter a in formula (2.13) and the relative error in the frequencies Δ_N are shown in the lower two rows of Table 1. In all cases, the value of the coefficient a varies over a narrow range (from 0.10 to 0.15) and decreases as the sphere radius increases.

Table 1

Tb, S	R = 1.55 cm		2.25		3.35	
	a	Δ_N, %	a	Δ_N, %	a	Δ_N, %
5.7 ± 0.1	0.13	14	–	–	–	–
9.9 ± 0.1	0.15	4	0.13	10	0.10	0.3
13.3 ± 0.1	0.14	3.3	0.13	2.3	0.12	5.5

At long times, the oscillation frequencies of the sphere are greater than the buoyancy period by several percent. The error in the frequencies varies irregularly, depending on the experimental conditions.

The values of the calculated period of the oscillations of a sphere in a viscous fluid are considerably less than in an ideal medium where it is $T_i = \sqrt{6}/n$ (Ref. 16, p. 135). The value of the ratio $T_i/T_b \approx 1.225$ considerably exceeds the errors in both the calculations and the measurements, which is indicates of the limitation on the possibility of using an ideal fluid model in problems of the free oscillations of bodies.

As the buoyancy period decreases, the influence of non-wave effects becomes greater and the value of Δ_N, which characterizes the difference between the oscillation frequencies and the buoyancy frequencies correspondingly increases (see the first column of Table 1).

A comparison of the experimental data on the oscillations of a sphere of radius R = 1.55 cm in fluids for different buoyancy periods, shown by the points in Fig. 2, with the data calculated using formula (2.13) (the solid, dashed and dot-dash curves) shows that stratification only slightly affects the body motion at the stage of free-fall of the body and the depth of the first immersion. However, it does affect the relative

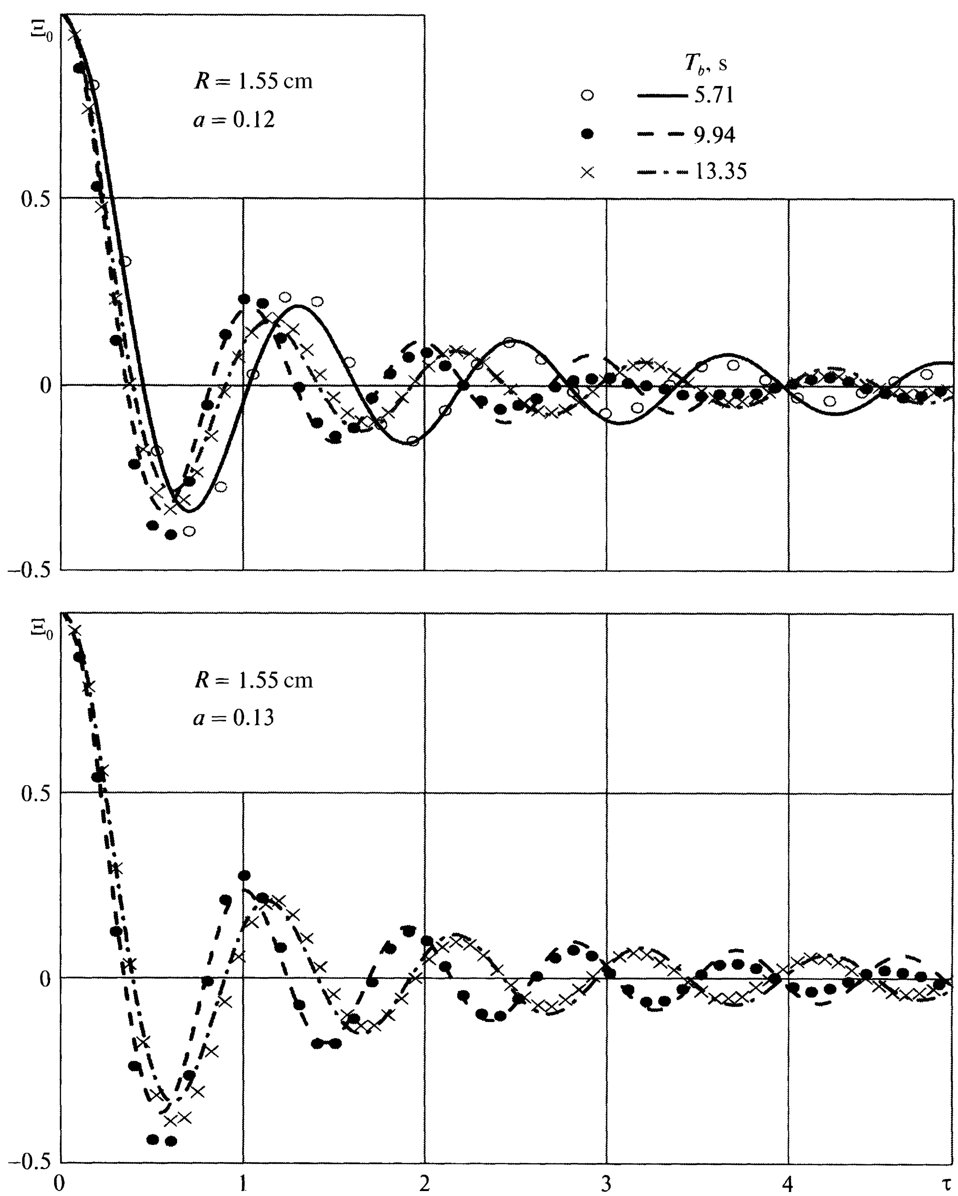

Fig. 2.

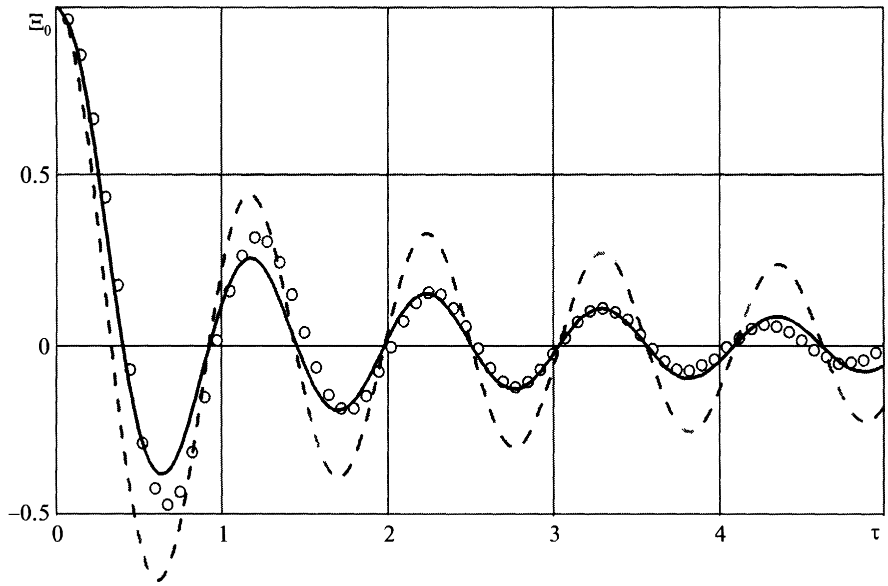

Fig. 3.

period of the oscillations which decreases as the buoyancy frequency increases (Fig. 2). This effect decreases as the sphere radius increases (the lower part of Fig. 2).

The comparative accuracy of the calculations of the oscillations of a sphere of radius $R = 3.35$ cm for $T_b = 13.35$ s using formula (2.13) with $a = 0.12$ (the solid curve) and the previously proposed technique[1] (the dashed curve) is illustrated in Fig. 3; the experimental results are shown by the points. The technique, based on the theory of internal waves in an ideal fluid,[1,2] gives oscillation periods in agreement with experiment, but the amplitude values are considerably overestimated and the rate of attenuation of the oscillations is underestimated. Formula (2.13) gives values of the depth of the first immersion and heights of the first ascent of the sphere which are too low by 10 - 15%. The trajectory of the motion of the centre of mass of a sphere is subsequently described by expression (2.13) within the limits of experimental error.

The method of calculation and the formulae obtained can be recommended for estimating periods of oscillation of neutral buoyancy spheres drifting in a stratified atmosphere or ocean.

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References

1. Larseb LH. Oscillatuibs of a neutrally buoyant sphere in a stratified fluid. *Deep Sea RE* 1969;**16**(6):587–603.
2. Lai RYS, Lee C-M. Addes mass of a spheroid oscillating in a linearly stratified fluid. *Intern J Engng Sci* 1981;**19**(11):1411–20.
3. Akulenko LD, Nesterov SV. Oscillations of a rigid body at the interface of two fluids. *Izv Akad Nauk SSSR MTT* 1987;**5**:34–40.
4. Chashechkin YuD, Levitskii VV. Hydrodynamics of the free oscillations of a sphere at the neutral buoyancy level in a continuously stratified fluid. *Dokl Russ Akad Nauk* 1999;**364**(1):52–6.
5. Cairns J, Munk W, Winant C. On the dynamics of neutrally buoyant capsules; an experimental drop in Lake Tahoe. *Deep Sea Res* 1979;**26A**(4):369–81.
6. Pylnev YuV, Razumeenko YuV. Investigations of the damped oscillations of a deeply immersed specially shaped buoy in a homogeneous and stratified fluid. *Izv Akad Nauk SSSR, MTT* 1991;**4**:71–9.
7. Prikhod'ko YuV, Chashechkin YuD. Hydrodynamics of the free oscillations on neutrally buoyant bodies at a depth on a continuously stratified fluid. *Izv Russ Akad Nauk MZhG* 2006;**4**:66–7.
8. Stretenskii LN. *Theory of Wave Motions*. Moscow: Nauka; 1977.
9. Landau LD, Lifshitz EM. *Fluid Dynamics*. Oxford: Pergamon; 1987.
10. Lomov SA. *Introduction to the General Theory of Singular Perturbations*. Moscow: Nauka; 1981.
11. Steveson TN, Bearon JN, Thomas NH. An internal wave in a viscous heat-conducting isothermal stmosphere. *J Fluid Mech* 1974;**65**(Pt 2):315–23.
12. Handbook of Mathematical Funcition with Formulas, Graphs and Mathematical Tables/Eds by M. Abramowitz and I. Stegun. Washington: Gov. Print, Off., 1964, 1979, 832c.
13. Nayfeh AH. *Introduction to Perturbation Techniques*. New York: Wiley; 1981, 519p.
14. Smirnov SA, Chashechkin YuD, Il'inykh YuS. A highly accurate method of measuring the buoyancy period profile. *Izmer Tekh* 1998;**6**:15–8.
15. Nalimov VV. *The Theory of Experiment*. Moscow: Nauka; 1971.
16. Brekhouskikh LM, Goncharov VV. *Introduction to Continuum Mechanies*. Moscow: Nauka; 1982.

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